The Lorentz Factor for the Buerger Precession Method

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nator drives its own decade directly and a special power unit has been designed to furnish power for the six numerator units.

AUXILIARY APPARATUS

The test gear necessary to set up this type of analyzer, to better than one part in four thousand, requires careful design. A Western Electric hydrogen mercury relay used as a sixty-cycle chopper was found to provide the most satisfactory method. The design provides an output pulse of forty volts, generated from a stabilized voltage and very accurate potentiometer, and shaped by resistance capacity networks to simulate the actual shape of pulse to be fed to the analyzer. The switch introduces negligible error provided the current taken by the contacts is of the order of five to ten milliamperes.

The accuracy of the analyzer can be increased by use of a "cut amplifier" which is inserted between the main amplifier and the analyzer. This unit consists of a single discriminator which can be set to any position within the amplitude spectrum, followed by an amplifier of some small gain such as three or four. The amplifier is designed to have good overload characteristics and a maximum output of forty-five volts. By this means a portion of the spectrum is spread over the whole analyzer range and all pulses outside the range are either missed or count in the top-most channel.

"Pulse stretchers" which peak rectify and extend the incoming pulses are used for millimicrosecond work and complete the facilities needed by the nuclear research worker.

BIBLIOGRAPHY TO PARTS IV AND V

(1) C. H. Wescott and G. C. Hanna, Rev. Sci. Instr. **20**, 181–188 (1949).
(2) H. F. Freundlich, E. P. Hinks, and W. J. Ozeroff, Rev. Sci. Instr. **18**, 90 (1947).
(3) W. E. Glenn, Jr., Nucleonics **4**, 50 (1949).
(4) E. H. Cooke-Yarborough, J. Bradwell, C. D. Florida, and G. A. Howells, Proc. Inst. Elec. Eng., Part III **97**, 108 (1950).
(5) W. C. Elmore and M. L. Sands, *Electronics: Experimental Techniques* (McGraw-Hill Book Company, Inc., New York, 1949).

CONCLUSION

The five related papers represent some three years' efforts directed to the realization of a comprehensive counting system for nuclear research.

The authors believe that this work represents the first of its kind, bringing to laboratory instruments the engineering practices necessary for reliability and long life with the full versatility afforded by electronic techniques. It is their personal conviction that the attainment of long-life techniques has implications far wider than the scope of this paper, for reliability is an essential corollary to the acceptance of electronic instrumentation in many fields, civil aviation and industrial process control representing but two examples.

ACKNOWLEDGMENTS

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The turntable described in Part IV was built to the design of Dr. A. J. Cipriani.

Mention must also be made of the enthusiastic cooperation and painstaking attention to details paid by Messrs. T. V. Sweeny and D. W. Bond of the Canadian Marconi Company, who were the development contractors for the various instruments

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The Lorentz factor for the Buerger precession method is derived. It turns out that the angular velocity of the motion of the reciprocal lattice through the sphere of reflection is not uniform as hitherto accepted. The Lorentz factor thus depends explicitly on all three cylindrical coordinates ξ, ζ, τ of the reciprocal lattice point under consideration and not on only as ξ and ζ.

In the versatile new precession method introduced by M. J. Buerger[1,2] for the recording of x-ray diffraction spots of single crystals the normal to the reciprocal lattice net plane being photographed (called n-level hereafter) moves on the surface of a circular cone of half-angle $\bar{\mu}$.[3] During this motion successive reciprocal lattice points of the n-level dip into the sphere of reflection while the corresponding net planes of the crystal go through their reflecting positions.

The right half of Fig. 4 shows a side view of the

[1] M. J. Buerger, *The Photography of the Reciprocal Lattice* (Am. Soc. for X-ray and Electron Diffraction, 1944).
[2] Evans, Tilden, and Adams, Rev. Sci. Instr. **20**, 155 (1949).
[3] In the main the symbols employed are those used by M. J. Buerger, *X-RayC rystallography* (John Wiley and Sons, Inc., New York, 1942) and reference 1.

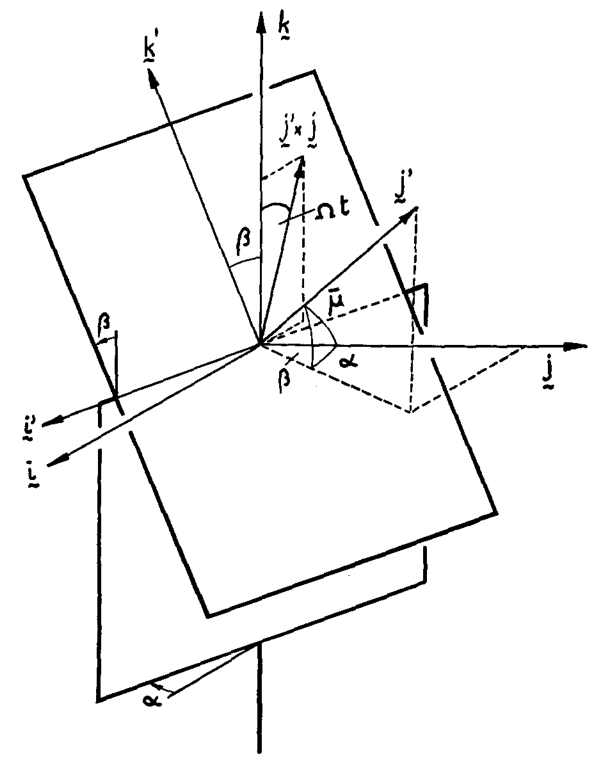

FIG. 1. Geometry of crystal suspension in two-axis universal joint.

sphere of reflection (of unit radius) being cut by the 0-level and the n-level of the reciprocal lattice. In analyzing the effects of the precession motion upon the appearance of the diffraction photograph, I have found it useful to take the reciprocal lattice as the fixed reference frame rather than the sphere of reflection. The sphere of reflection then carries out a precessional motion around the normal $O\,O'$ to the n-level; but since the sphere of reflection is structureless, this precession may be replaced by a simple rotation of the sphere around the line $O\,O'$. On closer analysis it is found, however, that for the customary construction of the precession goniometer[1] the angular velocity ω of this rotation is not constant but depends on the actual position in space of the precessing crystallographic axis.

Figure 1 shows the geometry of the crystal suspension by the two-axis universal joint of Buerger's instrument. The inclined plane represents the equatorial net plane of the reciprocal lattice. The vectors shown define two cartesian coordinate systems, one (unprimed) fixed in space and one (primed) tied to the precessing crystal. We are interested in the motion of the sphere of reflection relative to the second system, since this motion determines the interval during which each crystal reflection lights up as the surface of the sphere of reflection passes through the corresponding reciprocal lattice point. The equations of motion of the sphere in this system will be derived by first setting up equations describing the crystal motion in the unprimed system and later performing a transformation to the primed system.

Let then $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be unit vectors pointing along the x, y, z axes of a cartesian system such that $\mathbf{j}$ is in the direction of the incident x-ray beam and $\mathbf{k}$ along the vertical axis of the universal joint. Let $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ be unit vectors in a cartesian system x', y', z' tied to the reciprocal lattice, such that $\mathbf{i}'$ is in the direction of the horizontal axis of the universal joint and $\mathbf{j}'$ is perpendicular to the reciprocal lattice plane being photographed, enclosing the precession angle $\bar{\mu}$ with $\mathbf{j}$.

The transformation from one coordinate system to the other is given by the equations,

$$\begin{aligned}\mathbf{i}' &= \mathbf{i}\cos\alpha - \mathbf{j}\sin\alpha,\\ \mathbf{j}' &= \mathbf{i}\sin\alpha\cos\beta + \mathbf{j}\cos\alpha\cos\beta + \mathbf{k}\sin\beta,\\ \mathbf{k}' &= -\mathbf{i}\sin\alpha\sin\beta - \mathbf{j}\cos\alpha\sin\beta + \mathbf{k}\cos\beta,\end{aligned} \tag{1}$$

where the angles α and β are defined as in Fig. 1 and are related to $\bar{\mu}$ by the cosine law,

$$\cos\bar{\mu} = \cos\alpha\cos\beta. \tag{2}$$

To find the dependence of the angles α and β on the precessional motion we observe that the vector $\mathbf{j}'$ encloses the constant angle $\bar{\mu}$ with $\mathbf{j}$ around which it carries out a precession of constant angular velocity Ω. The vector product of $\mathbf{j}$ and $\mathbf{j}'$ is therefore a vector of length $|\mathbf{j}\times\mathbf{j}'| = \sin\bar{\mu}$ which rotates in the xz plane with the same angular velocity Ω. It is convenient to choose the time origin and the sense of rotation such that this motion is described by the equation,

$$\mathbf{j}'\times\mathbf{j} = (-\mathbf{i}\sin\Omega t + \mathbf{k}\cos\Omega t)\sin\bar{\mu}. \tag{3}$$

On the other hand, from (1) it follows that

$$\mathbf{j}'\times\mathbf{j} = -\sin\beta\,\mathbf{i} + \sin\alpha\cos\beta\,\mathbf{k}. \tag{4}$$

Combining (3) and (4) results in the desired equations for α and β:

$$\sin\beta = \sin\bar{\mu}\sin\Omega t, \tag{5}$$

$$\sin\alpha\cos\beta = \sin\bar{\mu}\cos\Omega t, \tag{6}$$

from which further

$$\cos\beta = (1-\sin^2\bar{\mu}\sin^2\Omega t)^{\frac{1}{2}}, \tag{7}$$

$$\sin\alpha = \sin\bar{\mu}\cos\Omega t(1-\sin^2\bar{\mu}\sin^2\Omega t)^{-\frac{1}{2}}, \tag{8}$$

$$\cos\alpha = \cos\bar{\mu}(1-\sin^2\bar{\mu}\sin^2\Omega t)^{-\frac{1}{2}}. \tag{9}$$

The sphere of reflection (of unit radius) is stationary in the unprimed coordinate system. Its center has the coordinates,

$$x_0 = z_0 = 0, \quad y_0 = -1. \tag{10}$$

The equations of motion of this center in the system tied to the reciprocal lattice are found by expressing the coordinates (10) in terms of primed coordinates. The transformation equations (1) yield together with (2)

$$x_0' = \sin\alpha, \quad y_0' = -\cos\alpha\cos\beta = -\cos\bar{\mu}, \quad z_0' = \cos\alpha\sin\beta. \tag{11}$$

The meaning of these equations is analyzed most simply by introducing cylindrical reciprocal lattice coordinates ξ, ζ, τ defined by

$$x' = \xi\cos\tau, \quad y' = \zeta, \quad z' = \xi\sin\tau. \tag{12}$$

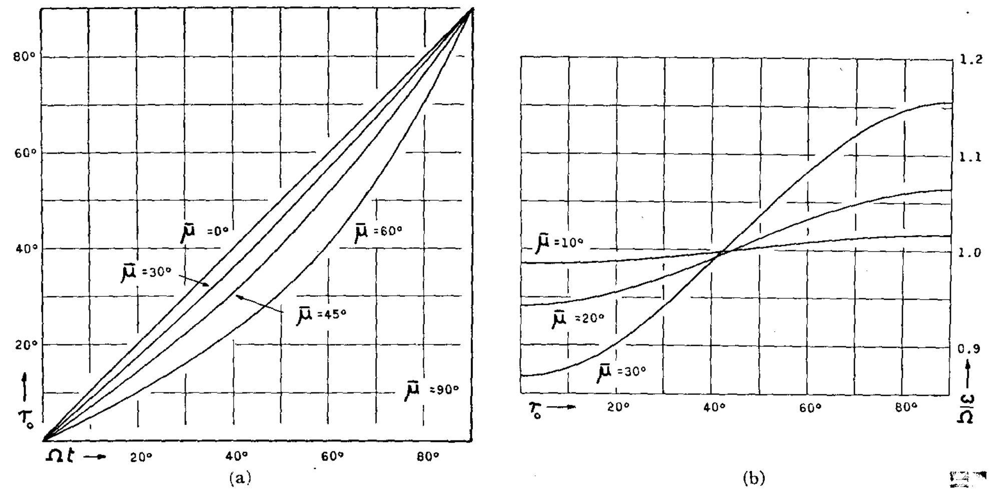

FIG. 2. (a) Time dependence of angular coordinate τ_0 of sphere of reflection in reciprocal space. (b) Variation of angular velocity ω with τ_0.

By comparison of (12) and (11) and using (5), (8), and (9) the cylindrical coordinates ξ_0 and ζ_0 of the center of the sphere of reflection are found to obey the conditions,

$$\xi_0=(x_0'+y_0'^2)^{\frac{1}{2}}=(\sin^2\alpha+\cos^2\alpha\,\sin^2\beta)^{\frac{1}{2}}=\sin\bar{\mu},$$

$$\zeta_0=-\cos\bar{\mu} \qquad (13)$$

which describe a circle of radius $\sin\bar{\mu}$ in the plane $\zeta=-\cos\bar{\mu}$. The coordinate τ_0 is in the same way found to satisfy the equation,

$$\tan\tau_0=z_0'/x_0'=\cot\alpha\,\sin\beta=\cos\bar{\mu}\,\tan\Omega t. \qquad (14)$$

The sphere of reflection thus moves on a circle in the reciprocal lattice; but τ_0, the angle which its center has covered at time t, is a rather involved function of Ωt, the angle through which the precessing crystallographic axis $\mathbf{j}'$ has progressed, and only in the trivial limit $\bar{\mu}=0$ are the two angles equal at all times. The angular velocity ω of the revolving sphere is given by

$$\omega=\dot{\tau}_0=\frac{\Omega\cos\bar{\mu}}{1-\sin^2\bar{\mu}\,\sin^2\Omega t}=\Omega\cos\bar{\mu}[1+\sin^2\tau_0\tan^2\bar{\mu}] \qquad (15)$$

$$=\omega(\tau_0,\bar{\mu}).$$

Figure 2 shows the dependence of τ_0 on Ωt and of ω on τ_0 for various values of the precession angle $\bar{\mu}$. For $\bar{\mu}=30°$ the deviation of ω from its average value Ω may amount to as much as 15 percent. The sphere attains maximum speed in its motion through the reciprocal lattice whenever the precession axis passes through the yz plane ($\Omega t=\pi/2$, $3\pi/2$), minimum speed whenever this axis passes through the xy plane ($\Omega t=0$, π). In the limiting case of $\bar{\mu}=90°$ the motion becomes discontinuous, the variable τ_0 jumping from 0 to π when $\Omega t=\pi/2$ and from π to 2π when $\Omega t=3\pi/2$, remaining constant in between. Indeed it was an analysis of this limiting case which started the present investigation.

In order to obtain the Lorentz factor it is necessary to find the velocity with which the sphere of reflection passes through a reciprocal lattice point $P(\xi,\zeta,\tau)$. Figure 3[4] shows the two positions of the sphere for which this passage occurs. One of these two positions is also represented in the left half of Fig. 4, which is a top view of the n-level, showing further the projection of the intersection of the 0-level with the sphere of reflection, and the annular space of the n-level swept out by the sphere of reflection as it rotates with angular velocity $\omega(\tau_0,\bar{\mu})$. The right half of Fig. 4 is a side view of the same situation.

The angles τ_0 describing the two passage positions of the center of the sphere of reflection are given by the relations,

$$\tau_0'=\tau-\eta,\quad \tau_0''=\tau+\eta \qquad (16)$$

and

$$\cos\eta=\frac{\xi^2+\sin^2\bar{\mu}-\sin^2\bar{\nu}}{2\xi\,\sin\bar{\mu}},\quad \sin\eta=\frac{\sin\bar{\nu}\,\sin\Upsilon}{\xi}, \qquad (17)$$

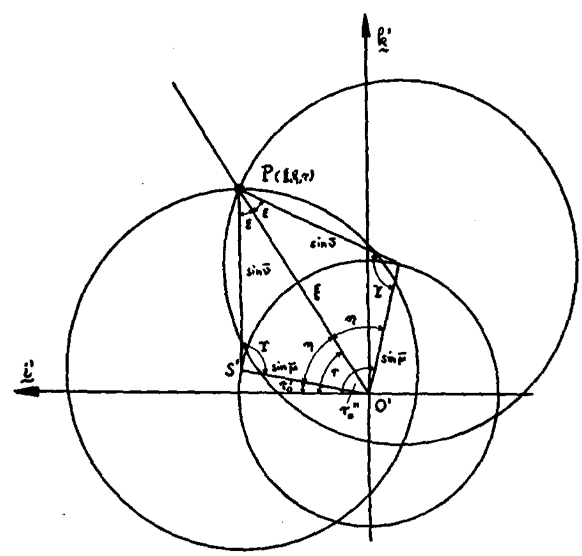

FIG. 3. Passage of sphere of reflection through reciprocal lattice point $P(\xi,\zeta,\tau)$.

[4] The author is indebted to Professor W. N. Lipscomb and Mr. W. J. Dulmage for pointing out an error in a previous form of Fig. 3 and of (17).

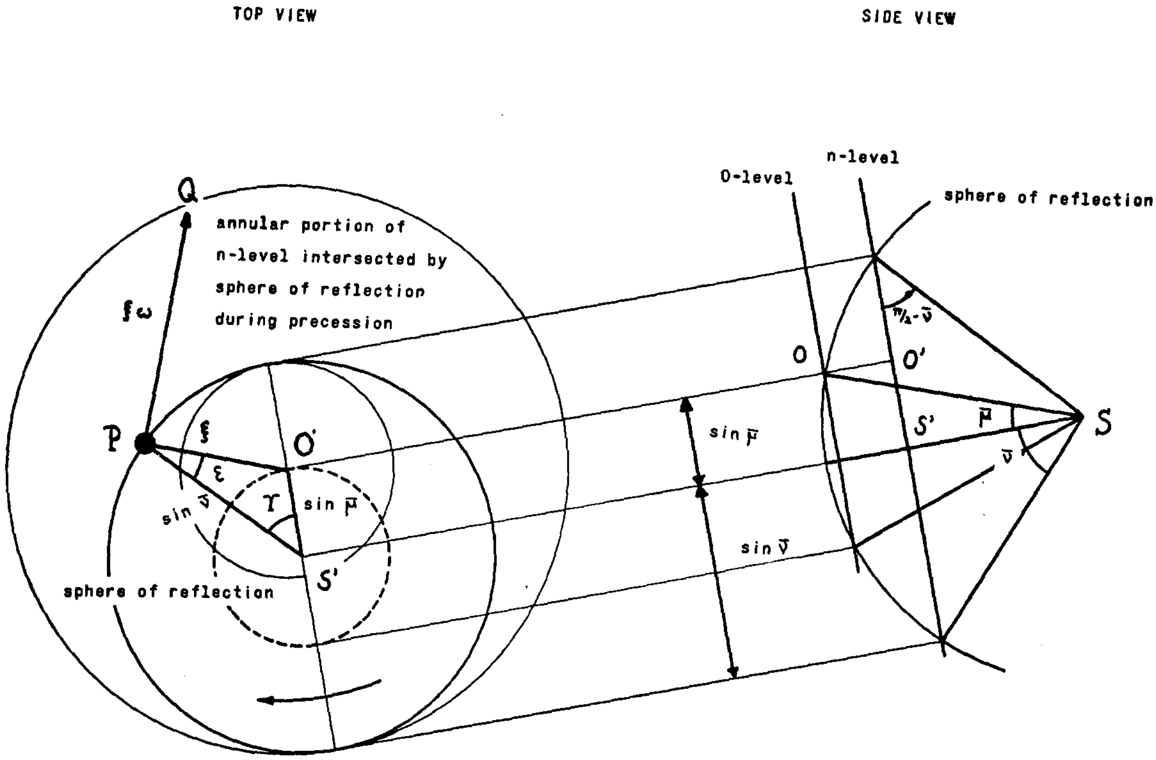

FIG. 4. Rotation of sphere of reflection in reciprocal lattice space.

the latter two resulting from application of the cosine and the sine theorem on triangle $PO'S'$ in Fig. 3.

The Lorentz factor L is inversely proportional to the radial component v_r of the velocity $\xi\omega(\tau_0, \bar{\mu})$ with which the sphere of reflection passes through $P(\xi, \zeta, \tau)$.[5] Let Υ and ϵ be defined as in Fig. 4 and let ψ be the angle between the radius SP of the sphere of reflection and the velocity vector PQ. Consider the right spherical triangle formed by the directions PS, PS', and PQ in which $<SPS'=\pi/2-\bar{\nu}$, $<S'PQ=\pi/2+\epsilon$, $<SPQ=\pi-\psi$, and dihedral $<(SPS'), (S'PQ)=\pi/2$. Then

$$\cos(\pi-\psi)=\cos(\tfrac{1}{2}\pi-\bar{\nu})\cos(\tfrac{1}{2}\pi+\epsilon)$$

or

$$\cos\psi=\sin\nu\sin\epsilon. \tag{18}$$

By application of the sine and cosine theorems to triangle $PS'O'$ the further relations,

$$\xi\sin\epsilon=\sin\bar{\mu}\sin\Upsilon, \tag{19}$$

$$\cos\Upsilon=(\sin^2\bar{\mu}+\sin^2\bar{\nu}-\xi^2)/2\sin\bar{\mu}\sin\bar{\nu} \tag{20}$$

are derived. Combining (18) and (19) with

$$v_r=\xi\omega\cos\psi=\omega(\tau_0, \bar{\mu})\sin\bar{\nu}\sin\bar{\mu}\sin\Upsilon, \tag{21}$$

the Lorentz factor is obtained,

$$L(\xi, \zeta, \tau)\sim\frac{1}{v_r(\tau_0')}+\frac{1}{v_r(\tau_0'')}=\frac{1}{\Omega\sin\Upsilon\sin\bar{\nu}\sin\bar{\mu}\cos\bar{\mu}}\times\left[\frac{1}{1+\tan^2\bar{\mu}\sin^2(\tau+\eta)}+\frac{1}{1+\tan^2\bar{\mu}\sin^2(\tau-\eta)}\right], \tag{22}$$

in which Υ and η may be expressed in terms of more convenient variables by the use of (17) and (20).

If ω were equal to Ω at all times, (22) would simplify to (see 20)

$$L(\xi, \zeta)\sim\frac{2}{\Omega\sin\bar{\mu}\sin\bar{\nu}\sin\Upsilon}, \tag{23}$$

which is identical with the usual form of the Lorentz factor for the Bragg case at oblique incidence, taking into account that $\bar{\mu}=\frac{1}{2}\pi-\mu$ and $\bar{\nu}=\frac{1}{2}\pi-\nu$ in customary notation.[3] This is not unexpected, since in a reference system tied to the reciprocal lattice the motion of the sphere of reflection is essentially the same for precession as for ordinary rotation. However, to insure constancy of ω a crystal and film suspension having "cylindrical symmetry" or a mechanical device involving, e.g., a special cam would be necessary.

For the 0-level ($\bar{\nu}=\bar{\mu}$) (22) takes the explicit form,

$$L(\xi, 0, \tau)\sim\frac{2}{\Omega\xi\cos\bar{\mu}\,[\sin^2\bar{\mu}-(\tfrac{1}{2}\xi)^2]^{\frac{1}{2}}}\,\frac{1+\tan^2\bar{\mu}\,[\cos^2\tau-(\tfrac{1}{2}\xi\sin\bar{\mu})^2\cos2\tau]}{1+2\tan^2\bar{\mu}\,[\cos^2\tau-(\tfrac{1}{2}\xi\sin\bar{\mu})^2\cos2\tau]+\tan^4\bar{\mu}\,[\cos^2\tau-(\tfrac{1}{2}\xi\sin\bar{\mu})^2]^2} \tag{24}$$

in which Υ and η have been eliminated by (17) and (20). A numerical evaluation of (24) for $\bar{\nu}=\bar{\mu}=30°$ is planned for a subsequent communication.

[5] M. J. Buerger, Proc. Nat. Acad. Sci. U.S. **26**, 637 (1940).